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# Propagator for a spin–Bose system with the Bose field coupled to a reservoir of harmonic oscillators

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**Abstract**

We consider the general problem of a single two-level atom interacting with a multimode radiation field (without the rotating-wave approximation), and additionally take the field to be coupled to a thermal reservoir. Using the method of bosonization of the spin operators in the Hamiltonian, and working in the Bargmann representation for all the boson operators, we obtain the propagator for the composite system using the techniques of functional integration, under a reasonable approximation scheme. The propagator is explicitly evaluated for a simplified version of the system with one spin and a dynamically coupled single-mode field. The results are also checked on the known problem of quantum Brownian motion.

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## 1. Introduction

The spin–Bose system of a two-level atom (or a spin-$1/2$ particle) interacting with a multimode quantum field serves as a useful model in a wide range of problems in atomic-optical and condensed matter physics. Most of the studies on decoherence and dissipation of a system of interest in contact with an environment are based on the spin–Bose model describing quantum Brownian motion of a simple harmonic oscillator in a harmonic-oscillator environment [1–4]. An interesting variant of the problem, that of measurement of the spin of a particle in the presence of an inhomogeneous magnetic field in a Stern–Gerlach apparatus when the particle is coupled to an environment of a collection of non-interacting harmonic oscillators via its position has also been worked out in detail [5–7]. The subtle issues of decoherence in the zero-temperature environment via statistical mixing or dephasing without energy dissipation have attracted much recent attention [4, 7]. The problems are usually treated using the Feynman–Vernon [1] influence functional formalism. The reservoir is assumed to be in thermal equilibrium at a temperature $T$ and the reservoir modes are eliminated to get the dynamics of the reduced density matrix for the system under consideration. Since the coupling constant for the interaction of the radiation with the atom is very small, in quantum optical applications, the rotating wave approximation (RWA) is used in the fully quantized treatment of the spin–Bose system. However, with this approximation, some interesting quantum electrodynamical effects, e.g., the zero-point fluctuations of the field and the atom, are missed out. Also, for many problems in condensed matter physics, for example, in the treatment of two-site small-polaron dynamics [8], the coupling constant is usually large and therefore the RWA is not quite appropriate for the description of the system.

In this paper we consider a more general problem of a single two-level atom interacting with a multimode radiation field which in turn is coupled to a thermal reservoir. This depicts, for example, the familiar spin–Bose system in a cavity with the boson field decaying through the walls of the cavity. In order to study the complete dynamics of the system, we will use a technically tractable and flexible method of functional integration developed by Papadopoulos [9] without making the RWA, and extend his work to obtain the propagator associated with our system.

A difficulty associated with handling path integrals for spins comes from the discrete matrix nature of the spin-Hamiltonians. To overcome this difficulty, the starting point of the work is to bosonize the Hamiltonian by representing the spin angular momentum operators in terms of boson operators following Schwinger's theory of angular momentum [10].

We then use the Bargmann representation [11] for all the boson operators. The Schrödinger representation of quantum states diagonalizes the position operator, expressing pure states as wavefunctions, whereas the Bargmann representation diagonalizes the creation operator $a^\dagger$, and expresses each state vector $|\psi\rangle$ in the Hilbert state $\mathcal{H}$ as an entire analytic function $f(\alpha)$ of a complex variable $\alpha$. The association $|\psi\rangle \longrightarrow f(\alpha)$ can be written conveniently in terms of the normalized coherent states $|\alpha\rangle$ which are the right eigenstates of the annihilator operator $a$:

$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle\alpha'|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha'|^2 - \tfrac{1}{2}|\alpha|^2 + \alpha'^*\alpha\right)$$

giving

$$f(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\langle\alpha^*|\psi\rangle.$$

In the Bargmann representation, the actions of $a$ and $a^\dagger$ are [9]

$$a^\dagger f(\alpha) = \alpha^* f(\alpha) \qquad a f(\alpha) = \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha^*}.$$

The propagator for the bosonized Hamiltonian in the expanded space is worked out as a path integral over coherent state variables. For a single spin–Bose system, the treatment of Papadopoulos [9] involves decoupling of the radiation variables from the coherent variables originating from the spin via a complex auxiliary field. The propagator is then expressed as a Gaussian functional integral over the auxiliary field of the product of two forced propagators, a radiation and a spin-bosonized propagator, and the propagator for the original system is extracted by appropriate projection.

The present paper is organized as follows. In section 2 we detail the method of bosonization and functional integration, and obtain the propagator for a Hamiltonian describing a single two-level atom interacting with a multimode radiation field which in turn is coupled to only one harmonic oscillator in the 'reservoir' to begin with. Two independent complex auxiliary fields are introduced to decouple the two interaction terms. In section 3 we generalize this result to obtain the propagator for the original Hamiltonian with a multimode thermal reservoir. In section 4 we present an application of our propagator for the problem of quantum Brownian motion. From our results the normalized reduced density matrix of the damped harmonic oscillator is reproduced satisfactorily. In section 5 we carry out an explicit evaluation of our propagator for a simplified version of the Hamiltonian with one spin and a dynamically coupled single-mode field. Finally, in section 6, we summarize the results.

## 2. Functional integration of the bosonized Hamiltonian

The Hamiltonian for our system of a single two-level atom interacting with a multimode radiation field coupled to a thermal reservoir is

$$H = H_F + H_A + H_{AF} + H_R + H_{FR} \tag{1}$$

where $H_F$, $H_A$, $H_R$ denote the Hamiltonians for the free field ($M$ modes), the atom and the reservoir ($K$ modes), respectively, $H_{AF}$ denotes the interaction of the atom with the electromagnetic field which is written in the dipole approximation without making the rotating wave approximation, $H_{FR}$ denotes the interaction of the field with the reservoir, for which we use the linear coupling model of the position–position kind:

$$H_F = \sum_{r=1}^{M} \hbar\Omega_r a_r^\dagger a_r \tag{2}$$

$$H_A = \frac{1}{2}\hbar\omega\sigma_z \tag{3}$$

$$H_R = \sum_{k=1}^{K} \hbar\overline{\Omega}_k b_k^\dagger b_k \tag{4}$$

$$H_{AF} = \sum_{r=1}^{M} \hbar\left(g_r^* a_r^\dagger + g_r a_r\right)\sigma_x \tag{5}$$

$$H_{FR} = \sum_{k=1}^{K}\sum_{r=1}^{M} \hbar g_{kr}\left(a_r^\dagger + a_r\right)\left(b_k^\dagger + b_k\right). \tag{6}$$

Here we have made use of the equivalence of a two-level atom and a spin-1/2 system, $\sigma_x$, $\sigma_z$ denote the standard Pauli spin matrices and are related to the spin-flipping (or atomic raising and lowering) operators $S_+$ and $S_-$: $\sigma_x = S_+ + S_-$, $\sigma_z = 2S_+S_- - 1$. In (2) $a_r^\dagger$, $a_r$ stand for the Bose creation and annihilation operators for the $M$ oscillators denoting the electromagnetic field, and in (4) $b_k^\dagger$, $b_k$ denote the Bose creation and annihilation operators for the $K$ oscillators representing the reservoir. In (5), $g_r^*$ and $g_r$ are the coupling constants for interaction of the field with the spin. In (6) $g_{kr}$ stands for the coupling constants (assumed real) for the interaction of the field with the reservoir.

We now outline the method of bosonization and functional integration for the propagator of the following simple part of the Hamiltonian (1) with only one harmonic oscillator in the ‘reservoir’:

$$H_1 = \sum_{r=1}^{M} \hbar\Omega_r a_r^\dagger a_r + \frac{1}{2}\hbar\omega\sigma_z + \sum_{r=1}^{M} \hbar\left(g_r^* a_r^\dagger + g_r a_r\right)\sigma_x + \hbar\overline{\Omega} b^\dagger b + \sum_{r=1}^{M} \hbar g_{1r}\left(a_r^\dagger + a_r\right)(b^\dagger + b) \tag{7}$$

where $g_{1r}$ are real.

In order to express the spin angular momentum operators in terms of boson operators, we employ Schwinger’s theory of angular momentum [10] by which any angular momentum can be represented in terms of a pair of boson operators with the usual commutation rules. The spin operators $\sigma_z$ and $\sigma_x$ can be written in terms of the boson operators $a_\beta$, $a_\beta^\dagger$ and $a_\gamma$, $a_\gamma^\dagger$ as

$$\sigma_z = a_\gamma^\dagger a_\gamma - a_\beta^\dagger a_\beta \qquad \sigma_x = a_\gamma^\dagger a_\beta + a_\beta^\dagger a_\gamma.$$

In the Bargmann representation [11] we associate the variable $\beta^*$ with the spin-down state and the variable $\gamma^*$ with the spin-up state. The flipping operators $S_-$ and $S_+$ take the forms $\beta^*\frac{\partial}{\partial\gamma^*}$ and $\gamma^*\frac{\partial}{\partial\beta^*}$, respectively, and we have

$$\sigma_z \longrightarrow \left(\gamma^*\frac{\partial}{\partial\gamma^*} - \beta^*\frac{\partial}{\partial\beta^*}\right) \qquad \sigma_x \longrightarrow \left(\gamma^*\frac{\partial}{\partial\beta^*} + \beta^*\frac{\partial}{\partial\gamma^*}\right).$$

Thus the bosonized form of the Hamiltonian (7) is

$$\begin{aligned} H_{B1} = {} & \sum_{r=1}^{M} \hbar\Omega_r\alpha_r^*\frac{\partial}{\partial\alpha_r^*} + \frac{1}{2}\hbar\omega\left(\gamma^*\frac{\partial}{\partial\gamma^*} - \beta^*\frac{\partial}{\partial\beta^*}\right) \\ & + \sum_{r=1}^{M}\hbar\left(g_r^*\alpha_r^* + g_r\frac{\partial}{\partial\alpha_r^*}\right)\left(\gamma^*\frac{\partial}{\partial\beta^*} + \beta^*\frac{\partial}{\partial\gamma^*}\right) + \hbar\overline{\Omega}\theta^*\frac{\partial}{\partial\theta^*} \\ & + \sum_{r=1}^{M}\hbar g_{1r}\left(\alpha_r^* + \frac{\partial}{\partial\alpha_r^*}\right)\left(\theta^* + \frac{\partial}{\partial\theta^*}\right) \end{aligned} \tag{8}$$

where $\theta^*$, $\partial/\partial\theta^*$ are the Bargmann representations for $b^\dagger$ and $b$, respectively, and $\alpha_r^*$, $\partial/\partial\alpha_r^*$ for $a_r^\dagger$ and $a_r$, respectively.

A particular solution of the Schrödinger equation with this bosonized Hamiltonian has the form [9]

$$U_1 = U_{00}\beta^*\beta' + U_{01}\beta^*\gamma' + U_{10}\gamma^*\beta' + U_{11}\gamma^*\gamma' \tag{9}$$

where the amplitudes $U_{ij}$ are now functions of time as well as the coherent state variables associated with the boson oscillators, with the initial condition

$$U_{ij}(t=0) = \exp(\theta^*\theta')\exp\left\{\sum_{r=1}^{M}\alpha_r^*\alpha_r'\right\}\delta_{ij} \qquad (i, j = 0, 1). \tag{10}$$

The initial condition for the expanded propagator associated with the bosonized Hamiltonian given by (8) is then given as

$$U(t=0) = \exp(\theta^*\theta')\exp\left\{\sum_{r=1}^{M}\alpha_r^*\alpha_r'\right\}\exp\{\beta^*\beta' + \gamma^*\gamma'\}. \tag{11}$$

If the Hamiltonian is in the normal form given by $H(\alpha^*, \partial/\partial\alpha^*, t)$, the associated propagator is given as a path integral over coherent state variables as [12]

$$U(\alpha^*, t; \alpha', 0) = \int \mathbf{D}^2\{\alpha\}\exp\left\{\sum_{0\leqslant\tau<t}\alpha^*(\tau+)\alpha(\tau) - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau\, H(\alpha^*(\tau+), \alpha(\tau), \tau)\right\}. \tag{12}$$

Here $\sum_{0\leqslant\tau<t}\alpha^*(\tau+)\alpha(\tau)$ stands for $\sum_{j=0}^{N-1}\alpha^*(\tau_{j+1})\alpha(\tau_j)$ in the subdivision of the interval $[0, t]$, i.e., when $\tau$ stands for $\tau_j$, $\tau+$ stands for the next point $\tau_{j+1}$ in the subdivision. Similarly, in the subdivision scheme,

$$\int_0^t \mathrm{d}\tau\, H(\alpha^*(\tau+), \alpha(\tau), \tau) = \sum_{j=0}^{N-1} H(\alpha^*(\tau_{j+1}), \alpha(\tau_j), \tau_j)\Delta\tau_j.$$

Also, the path differential in (12) is

$$\mathbf{D}^2\{\alpha\} = \prod_{0<\tau<t} D^2\alpha(\tau) \tag{13}$$

where the weighted differential is

$$D^2\alpha(\tau) = \frac{1}{\pi}\exp(-|\alpha(\tau)|^2)\,\mathrm{d}^2\alpha(\tau). \tag{14}$$

Using (12) we can write the propagator for the bosonized Hamiltonian given by (8) as follows:

$$\begin{aligned}
&u_1(\theta^*, \boldsymbol{\alpha}^*, \beta^*, \gamma^*, t; \theta', \boldsymbol{\alpha}', \beta', \gamma', 0)\\
&\quad = \int \mathbf{D}^2\{\theta\}\mathbf{D}^2\{\boldsymbol{\alpha}\}\,\mathbf{D}^2\{\beta\}\,\mathbf{D}^2\{\gamma\}\exp\left\{\sum_{0\leqslant\tau<t}\left[\theta^*(\tau+)\theta(\tau) + \sum_{r=1}^{M}\alpha_r^*(\tau+)\alpha_r(\tau)\right.\right.\\
&\qquad \left. + \beta^*(\tau+)\beta(\tau) + \gamma^*(\tau+)\gamma(\tau)\right] - \mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ \Omega_r\alpha_r^*(\tau+)\alpha_r(\tau)\\
&\qquad - \mathrm{i}\int_0^t \mathrm{d}\tau\overline{\Omega}\theta^*(\tau+)\theta(\tau) - \mathrm{i}\frac{\omega}{2}\int_0^t \mathrm{d}\tau[\gamma^*(\tau+)\gamma(\tau) - \beta^*(\tau+)\beta(\tau)]\\
&\qquad - \mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau[g_r^*\alpha_r^*(\tau+) + g_r\alpha_r(\tau)][\gamma^*(\tau+)\beta(\tau) + \beta^*(\tau+)\gamma(\tau)]\\
&\qquad \left. - \mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ g_{1r}(\alpha_r^*(\tau+) + \alpha_r(\tau))(\theta^*(\tau+) + \theta(\tau))\right\}.
\end{aligned} \tag{15}$$

In the above equation $\boldsymbol{\alpha}$ is a vector with components $\{\alpha_r\}$, and $\mathbf{D}^2\{\boldsymbol{\alpha}\} = \prod_{r=1}^{M}\mathbf{D}^2\{\alpha_r\}$.

The propagator given by (15) is evaluated by the introduction of two independent complex auxiliary fields $f(\tau)$ and $f_1(\tau)$ which decouple the two interaction terms in (15) as follows:

$$\begin{aligned}
&\exp\left[-\mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau(g_r^*\alpha_r^*(\tau+) + g_r\alpha_r(\tau))(\gamma^*(\tau+)\beta(\tau) + \beta^*(\tau+)\gamma(\tau))\right]\\
&\quad = \int \mathbf{D}^2\{f\}\exp\left[-\mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ f^*(\tau)(g_r^*\alpha_r^*(\tau+) + g_r\alpha_r(\tau))\right]\\
&\qquad \times\exp\left[\int_0^t \mathrm{d}\tau\ f(\tau)(\gamma^*(\tau+)\beta(\tau) + \beta^*(\tau+)\gamma(\tau))\right]
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\exp\left[-\mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ g_{1r}(\alpha_r^*(\tau+) + \alpha_r(\tau))(\theta^*(\tau+) + \theta(\tau))\right]\\
&\quad = \int \mathbf{D}^2\{f_1\}\exp\left[-\mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ f_1^*(\tau)g_{1r}(\alpha_r^*(\tau+) + \alpha_r(\tau))\right]\\
&\qquad \times\exp\left[\int_0^t \mathrm{d}\tau\ f_1(\tau)(\theta^*(\tau+) + \theta(\tau))\right].
\end{aligned} \tag{17}$$

Here we have used the $\delta$-*functional* identity [9]

$$\int \mathbf{D}^2\{X\}P[X^*(t)]\exp\left\{\int_0^t \mathrm{d}\tau\ Y(\tau)X(\tau)\right\} = P[Y(t)] \tag{18}$$

where $\mathbf{D}^2\{X\}$ is a functional differential:

$$\mathbf{D}^2\{X\} = \exp\left(-\int_0^t \mathrm{d}\tau |X(\tau)|^2\right) \prod_{0\leqslant\tau<t} \left(\frac{\mathrm{d}\tau}{\pi}\right) \mathrm{d}^2X(\tau) \tag{19}$$

and $P[X^*(t)]$ is an explicit functional of $X^*$ only. The roles of $X^*$ and $X$ are interchangeable in the identity (18). Making use of (16) and (17) we can write the propagator given by (15) as

$$\begin{aligned} &u_1(\theta^*, \boldsymbol{\alpha}^*, \beta^*, \gamma^*, t; \theta', \boldsymbol{\alpha}', \beta', \gamma', 0) \\ &\quad = \int\int \mathbf{D}^2\{f\}\mathbf{D}^2\{f_1\} G_1(\boldsymbol{\alpha}^*, t; \boldsymbol{\alpha}', 0; [f^*, f_1^*]) \\ &\qquad \times M_1(\theta^*, t; \theta', 0; [f_1]) N_1(\beta^*, \gamma^*, t; \beta', \gamma', 0; [f]). \end{aligned} \tag{20}$$

Here $G_1$ stands for the propagator for

$$H_{G_1} = \hbar \sum_{r=1}^{M} \left[\Omega_r {\boldsymbol{\alpha}^*}_r \frac{\partial}{\partial\alpha_r^*} + (f^*(t)g_r^* + f_1^*(t)g_{1r})\alpha_r^* + (f^*(t)g_r + f_1^*(t)g_{1r})\alpha_r\right] \tag{21}$$

$M_1$ stands for the propagator for

$$H_{M_1} = \hbar\left[\overline{\Omega}\theta^* \frac{\partial}{\partial\theta^*} + \mathrm{i} f_1(t)(\theta^* + \theta)\right] \tag{22}$$

$N_1$ stands for the propagator for

$$H_{N_1} = \frac{\hbar\omega}{2}\left(\gamma^* \frac{\partial}{\partial\gamma^*} - \beta^* \frac{\partial}{\partial\beta^*}\right) + \mathrm{i}\hbar f(t)\left(\gamma^* \frac{\partial}{\partial\beta^*} + \beta^* \frac{\partial}{\partial\gamma^*}\right). \tag{23}$$

These obey the Schrödinger equations $\mathrm{i}\hbar\partial G_1/\partial t = H_{G_1}G_1, \mathrm{i}\hbar\partial M_1/\partial t = H_{M_1}M_1$ and $\mathrm{i}\hbar\partial N_1/\partial t = H_{N_1}N_1$ with the initial conditions:

$$\begin{aligned} &G_1(t=0) = \exp\left\{\sum_{r=1}^{M} \alpha_r^*\alpha_r'\right\} \qquad M_1(t=0) = \exp(\theta^*\theta') \\ &N_1(t=0) = \exp(\beta^*\beta' + \gamma^*\gamma'). \end{aligned} \tag{24}$$

Now the propagator $G_1$ is given by

$$G_1 = G_{1a}G_{1b}G_{1c} \tag{25}$$

where

$$\begin{aligned} G_{1a} = \exp\Bigg\{&\sum_{r=1}^{M} \alpha_r^*\alpha_r' \,\mathrm{e}^{-\mathrm{i}\Omega_r t} - \sum_{r=1}^{M}\Bigg[\mathrm{i}\alpha_r^* g_r^* \int_0^t \mathrm{d}\tau\, f^*(\tau)\,\mathrm{e}^{-\mathrm{i}\Omega_r(t-\tau)} + \mathrm{i}\alpha_r' g_r \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\Omega_r\tau} f^*(\tau) \\ &+ |g_r|^2 \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\Omega_r(\tau-\tau')} f^*(\tau) f^*(\tau')\Bigg]\Bigg\} \end{aligned} \tag{26}$$

$$\begin{aligned} G_{1b} = \exp\Bigg\{&-\sum_{r=1}^{M}\Bigg[\mathrm{i}\alpha_r^* g_{1r} \int_0^t \mathrm{d}\tau\, f_1^*(\tau)\,\mathrm{e}^{-\mathrm{i}\Omega_r(t-\tau)} + \mathrm{i}\alpha_r' g_{1r} \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\Omega_r\tau} f_1^*(\tau) \\ &+ (g_{1r})^2 \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\Omega_r(\tau-\tau')} f_1^*(\tau) f_1^*(\tau')\Bigg]\Bigg\} \end{aligned} \tag{27}$$

$$G_{1c} = \exp\left\{-\sum_{r=1}^{M}\left[g_r g_{1r}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\Omega_r(\tau-\tau')} f^*(\tau) f_1^*(\tau')\right.\right.$$
$$\left.\left. + g_{1r} g_r^* \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}\Omega_r(\tau-\tau')} f_1^*(\tau) f^*(\tau')\right]\right\}. \tag{28}$$

The propagator $M_1$ is given by

$$M_1 = \exp\{\theta^*\theta'\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t}\}\exp\left\{\int_0^t \mathrm{d}\tau\, \widetilde{a}(\tau) f_1(\tau)\right\}\exp\left\{\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau''\, \mathrm{e}^{-\mathrm{i}\overline{\Omega}(\tau-\tau'')} f_1(\tau) f_1(\tau'')\right\} \tag{29}$$

where

$$\widetilde{a}(\tau) = \theta^*\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}(t-\tau)} + \theta'\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}\tau}. \tag{30}$$

In order to facilitate the functional averaging we make an approximation by keeping only the symmetrical part of the two-time kernel in the RHS of (29):

$$M_1 \longrightarrow M_1^0 = \exp\{\theta^*\theta'\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t}\}\exp\left\{\int_0^t \mathrm{d}\tau\, \widetilde{a}(\tau) f_1(\tau)\right\}$$
$$\times \exp\left\{\frac{1}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau''\, \mathrm{e}^{-\mathrm{i}\overline{\Omega}(\tau-\tau'')} f_1(\tau) f_1(\tau'')\right\}. \tag{31}$$

The bilinear functional occurring on the RHS of (31) can be replaced by a linear functional in $f_1(\tau)$ with the help of a complex auxiliary variable $z$ as follows [14]:

$$\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'' a(\tau) b(\tau'') f_1(\tau) f_1(\tau'')\right]$$
$$= \int \frac{\mathrm{d}^2 z}{\mathrm{i}\pi}\exp\left[\frac{\mathrm{i}}{\sqrt{\hbar}}\int_0^t \mathrm{d}\tau (z a(\tau) + z^* b(\tau)) f_1(\tau) + \mathrm{i}|z|^2\right]. \tag{32}$$

With

$$a(\tau) = \frac{\mathrm{i}\hbar}{2}\mathrm{e}^{-\mathrm{i}\overline{\Omega}\tau} \qquad b(\tau) = \mathrm{e}^{\mathrm{i}\overline{\Omega}\tau} \tag{33}$$

the propagator $M_1^0$ can be written as

$$M_1^0 = \exp\{\theta^*\theta'\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t}\}\int \frac{\mathrm{d}^2 z}{\mathrm{i}\pi}\exp\{\mathrm{i}|z|^2\}\exp\left\{\int_0^t \mathrm{d}\tau\, \widetilde{e}(\tau) f_1(\tau)\right\} \tag{34}$$

where

$$\widetilde{e}(\tau) \equiv \widetilde{a}(\tau) + \frac{\mathrm{i}}{\sqrt{\hbar}}(z a(\tau) + z^* b(\tau)). \tag{35}$$

The propagator $N_1$ has been obtained by Papadopoulos [9] as

$$N_1 = \exp\{Q_{00}\beta^*\beta' + Q_{01}\beta^*\gamma' + Q_{10}\gamma^*\beta' + Q_{11}\gamma^*\gamma'\}$$
$$= \sum_{l=0}^{\infty}\frac{1}{l!}\left[(\beta^*, \gamma^*) Q \begin{pmatrix}\beta' \\ \gamma'\end{pmatrix}\right]^l. \tag{36}$$

Here the $l = 1$ term corresponds to the single spin case given by (7). The $Q$ are given in terms of a power series in the spin-flipping energy:

$$Q = \sum_{n=0}^{\infty} Q^{(n)} \tag{37}$$

with

$$Q^{(n)}(t) = \left(\frac{\mathrm{i}\omega}{2}\sigma_z\right)^n \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \cdots \int_0^{\tau_2} \mathrm{d}\tau_1 \times \exp\left[\sigma_x \left(\int_0^{\tau_1} - \int_{\tau_1}^{\tau_2} + \cdots + (-1)^n \int_{\tau_n}^{t}\right) \mathrm{d}\tau\, f(\tau)\right]. \tag{38}$$

Now we can write the propagator given in (20) for the bosonized Hamiltonian (8) approximately as

$$u_1 = \int \mathbf{D}^2\{f\} G_{1a} \left\{\int \mathbf{D}^2\{f_1\} G_{1b} G_{1c} M_1^0\right\} N_1 \tag{39}$$

with $G_{1a}, G_{1b}, G_{1c}$ given by (26), (27), (28), respectively, $M_1^0$ by (34), $N_1$ by (36). The propagator $\overline{u}_1$ corresponding to the Hamiltonian of (7) is given by $u_1$ in (39) with only the $l = 1$ term in $N_1$. By repeated use of the $\delta$-functional identity given by (18), the propagator $\overline{u}_1$ is obtained with its amplitudes $U_{ij}$ $(i, j = 0, 1)$ in a matrix arrangement as

$$\begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} = \exp\{\theta^*\theta' \,\mathrm{e}^{-\overline{\Omega}t}\} \exp\left\{\sum_{r=1}^{M} \alpha_r^* \alpha_r' \,\mathrm{e}^{-\mathrm{i}\Omega_r t}\right\} \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}\omega}{2}\right)^n \times \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1} \cdots \int_0^{\tau_2} \mathrm{d}\tau_1 \left\{\frac{K_{\widetilde{\alpha}}(\overline{\Omega})}{\sqrt{\widetilde{\alpha}^2 + 4\Theta\xi}} \exp\left\{F_2^{(n)}\right\} \exp\left\{-\frac{\mathrm{i}}{4}A\right\} \times \begin{bmatrix} \cosh(\overline{\phi}^{(n)}) & \sinh(\overline{\phi}^{(n)}) \\ (-1)^n \sinh(\overline{\phi}^{(n)}) & (-1)^n \cosh(\overline{\phi}^{(n)}) \end{bmatrix}\right\} \tag{40}$$

where

$$K_{\widetilde{\alpha}}(\overline{\Omega}) = \exp\{-m(\overline{\Omega})\theta^* \,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t} - m(-\overline{\Omega})\theta' - N(\overline{\Omega})(\theta^*)^2 \,\mathrm{e}^{-2\mathrm{i}\overline{\Omega}t} - N(-\overline{\Omega})(\theta')^2 - 2(\widetilde{\alpha} - 1)\theta^*\theta' \,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t}\} \tag{41}$$

$$F_2^{(n)} = -\sum_{r=1}^{M} \frac{|g_r|^2}{\Omega_r^2} \left\{2n + 1 - \mathrm{i}\Omega_r t + 2\sum_{k=1}^{n} (-1)^k \,\mathrm{e}^{-\mathrm{i}\Omega_r \tau_k} + (-1)^{n+1} \,\mathrm{e}^{-\mathrm{i}\Omega_r t} \times \left(1 + 2\sum_{k=1}^{n} (-1)^k \,\mathrm{e}^{\mathrm{i}\Omega_r \tau_k}\right) + 4\sum_{j=2}^{n} \sum_{k=1}^{j-1} (-1)^{j+k} \,\mathrm{e}^{-\mathrm{i}\Omega_r(\tau_j - \tau_k)}\right\} \tag{42}$$

$$m(\overline{\Omega}) = \sum_{r=1}^{M} \{\alpha_r^* g_{1r} \overline{S}_1(\overline{\Omega}) + \alpha_r' g_{1r} \overline{S}_2(\overline{\Omega})\} \tag{43}$$

$$\overline{S}_1(\overline{\Omega}) = \frac{1}{(\Omega_r + \overline{\Omega})} (\mathrm{e}^{\mathrm{i}\overline{\Omega}t} - \mathrm{e}^{-\mathrm{i}\Omega_r t}) \tag{44}$$

$$\overline{S}_2(\overline{\Omega}) = \frac{1}{(\Omega_r - \overline{\Omega})} (1 - \mathrm{e}^{-\mathrm{i}(\Omega_r - \overline{\Omega})t}) \tag{45}$$

$$N(\overline{\Omega}) = \sum_{r=1}^{M} (g_{1r})^2 \,\overline{t}(\overline{\Omega}) \tag{46}$$

$$\overline{t}(\overline{\Omega}) = \frac{1}{(\Omega_r + \overline{\Omega})} \left\{\frac{1}{2\overline{\Omega}} (1 - \mathrm{e}^{\mathrm{i}2\overline{\Omega}t}) + \frac{1}{(\Omega_r - \overline{\Omega})} (1 - \mathrm{e}^{-\mathrm{i}(\Omega_r - \overline{\Omega})t})\right\} \tag{47}$$

$$\widetilde{\alpha} = 1 + \tfrac{1}{2}(P(\overline{\Omega}) + P(-\overline{\Omega})) \tag{48}$$

$$P(\overline{\Omega}) = \sum_{r=1}^{M} (g_{1r})^2 g'(\overline{\Omega}) \tag{49}$$

$$g'(\overline{\Omega}) = \frac{1}{(\Omega_r - \overline{\Omega})} \left\{ -\mathrm{i}t + \frac{1}{(\Omega_r - \overline{\Omega})} (1 - \mathrm{e}^{-\mathrm{i}(\Omega_r - \overline{\Omega})t}) \right\} \tag{50}$$

$$\Theta = -\frac{\hbar}{4} N(-\overline{\Omega}) \tag{51}$$

$$\xi = \frac{1}{\hbar} N(\overline{\Omega}) \tag{52}$$

$$\overline{\phi}^{(n)} = \phi_{21}^{(n)} - n_1(\overline{\Omega})\theta^* \,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t} - n_1(-\overline{\Omega})\theta' - \frac{\mathrm{i}}{4} C \tag{53}$$

$$\phi_{21}^{(n)} = \sum_{r=1}^{M} \left( -\alpha_r^* \,\mathrm{e}^{-\mathrm{i}\Omega_r t} T_{2r}^{(n)*} + \alpha_r' T_{2r}^{(n)} \right) \tag{54}$$

$$T_{2r}^{(n)} = -\frac{g_r}{\Omega_r} \left\{ 1 + 2 \sum_{j=1}^{n} (-1)^j \,\mathrm{e}^{-\mathrm{i}\Omega_r \tau_j} + (-1)^{n+1} \,\mathrm{e}^{-\mathrm{i}\Omega_r t} \right\} \tag{55}$$

$$A = a_{11}\left(\widetilde{\beta}_1^2 + \widetilde{\beta}_2^2\right) + a_{12}(\widetilde{\gamma}_1 \widetilde{\beta}_1 + \widetilde{\gamma}_2 \widetilde{\beta}_2) + a_{21}(\widetilde{\beta}_1 \widetilde{\gamma}_1 + \widetilde{\beta}_2 \widetilde{\gamma}_2) + a_{22}\left(\widetilde{\gamma}_1^2 + \widetilde{\gamma}_2^2\right) \tag{56}$$

$$C = 2a_{11}\widetilde{\beta}_1 \widetilde{\beta}_2 + a_{12}(\widetilde{\gamma}_1 \widetilde{\beta}_2 + \widetilde{\gamma}_2 \widetilde{\beta}_1) + a_{21}(\widetilde{\beta}_1 \widetilde{\gamma}_2 + \widetilde{\beta}_2 \widetilde{\gamma}_1) + 2a_{22}\widetilde{\gamma}_1 \widetilde{\gamma}_2 \tag{57}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -\widetilde{\Theta} & \widetilde{\alpha}/2 \\ \widetilde{\alpha}/2 & -\widetilde{\xi}^{-1} \end{pmatrix} = \begin{pmatrix} -\widetilde{\xi}/D & -\widetilde{\alpha}/2D \\ -\widetilde{\alpha}/2D & -\widetilde{\Theta}/D \end{pmatrix} \tag{58}$$

$$\widetilde{\Theta} = \mathrm{i}\Theta \qquad \widetilde{\xi} = \mathrm{i}\xi \qquad \widetilde{\beta}_1 = \mathrm{i}\beta_1 \qquad \widetilde{\beta}_2 = \mathrm{i}\beta_2 \qquad \widetilde{\gamma}_1 = \mathrm{i}\gamma_1 \qquad \widetilde{\gamma}_2 = \mathrm{i}\gamma_2 \tag{59}$$

$$D = -\tfrac{1}{4}(\widetilde{\alpha}^2 + 4\xi\Theta) \tag{60}$$

$$\beta_1 = \sqrt{\hbar}\left[\tfrac{1}{2} m(-\overline{\Omega}) + N(-\overline{\Omega})\theta' + (\widetilde{\alpha} - 1)\theta^* \,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t}\right] \tag{61}$$

$$\beta_2 = \sqrt{\hbar}\left[\tfrac{1}{2} n_1(-\overline{\Omega})\right] \tag{62}$$

$$n_1(\overline{\Omega}) = \sum_{r=1}^{M} \left\{ g_r g_{1r} Y_2^{(n)}(\overline{\Omega}) + g_{1r} g_r^* Z_2^{(n)}(\overline{\Omega}) \right\} \tag{63}$$

$$Y_2^{(n)}(\overline{\Omega}) = \frac{1}{(\Omega_r + \overline{\Omega})} \left\{ (-1)^{n+1} \left[ \frac{\mathrm{e}^{\mathrm{i}\overline{\Omega}t}}{\overline{\Omega}} + \frac{\mathrm{e}^{-\mathrm{i}\Omega_r t}}{\Omega_r} \right] + \sum_{j=1}^{n} (-1)^j 2 \left( \frac{\mathrm{e}^{\mathrm{i}\overline{\Omega}\tau_j}}{\overline{\Omega}} + \frac{\mathrm{e}^{-\mathrm{i}\overline{\Omega}_r \tau_j}}{\Omega_r} \right) + \left( \frac{1}{\overline{\Omega}} + \frac{1}{\Omega_r} \right) \right\} \tag{64}$$

$$Z_2^{(n)}(\overline{\Omega}) = \frac{1}{\Omega_r}\left\{(-1)^{n+1}\left[\frac{\mathrm{e}^{\mathrm{i}\overline{\Omega}t}}{\overline{\Omega}} + (-1)^n \frac{\mathrm{e}^{-\mathrm{i}(\Omega_r-\overline{\Omega})t}}{(\Omega_r-\overline{\Omega})}\right] + \frac{2}{(\Omega_r-\overline{\Omega})}\sum_{j=1}^{n}(-1)^{j+1}\,\mathrm{e}^{-\mathrm{i}\Omega_r(t-\tau_j)}\,\mathrm{e}^{\mathrm{i}\overline{\Omega}t} - \frac{2\Omega_r}{\overline{\Omega}(\Omega_r-\overline{\Omega})}\sum_{j=1}^{n}(-1)^{j+1}\,\mathrm{e}^{\mathrm{i}\overline{\Omega}\tau_j} + \left(\frac{1}{\overline{\Omega}} + \frac{1}{(\Omega_r-\overline{\Omega})}\right)\right\} \tag{65}$$

$$\gamma_1 = -\frac{\mathrm{i}}{\sqrt{\hbar}}[m(\overline{\Omega}) + 2N(\overline{\Omega})\theta^*\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}t} + 2(\widetilde{\alpha}-1)\theta'] \tag{66}$$

$$\gamma_2 = \frac{-\mathrm{i}}{\sqrt{\hbar}}n_1(\overline{\Omega}). \tag{67}$$

Equation (40), together with (41)–(67), gives the propagator for the Hamiltonian of (7). Note that this is a power series in the spin-flipping energy. Equation (40) satisfies the initial condition given by (10). Also, it can easily be seen that (40) reduces to the propagator for the single spin–Bose system obtained by Papadopoulos [9] when the appropriate terms in the Hamiltonian in (7) are set to zero.

We have obtained the propagator $\overline{u}_1$ with the approximate expression $M_1^0$ given by (31) for $M_1$ given in (29). Let us now examine the validity of this approximation. If we scale the field–reservoir coupling constant $g_{1r}$ in the Hamiltonian (7) by writing $\overline{\gamma}g_{1r}$ in place of $g_{1r}$, we can redefine the auxiliary field $f_1$ as $\overline{\gamma}f_1$ (and $f_1^*$ as $\overline{\gamma}f_1^*$) and get back the same expressions for the propagators $G_{1b}$ and $G_{1c}$; then the last exponent in the propagator $M_1^0$ in (34) gets scaled with $\widetilde{e}(\tau)$ in (35) redefined as $\overline{\gamma}\widetilde{e}(\tau)$. The field–reservoir coupling is usually very weak, i.e., $\overline{\gamma} \ll 1$. Using the first cumulant approximation of the actual propagator, it can be seen that the approximate propagator $\overline{u}_1$ we have got gives a fairly accurate microscopic description of the system since the difference with the actual propagator comes out to be of the order of fourth power of the coupling constant.

## 3. Many-mode reservoir

Returning to the original problem with the Hamiltonian (1) with $K$ modes in the reservoir, the propagator for the bosonized Hamiltonian now becomes (cf (15) for the Hamiltonian (8))

$$\widetilde{U}_1(\boldsymbol{\theta}^*, \boldsymbol{\alpha}^*, \beta^*, \gamma^*, t; \boldsymbol{\theta}', \boldsymbol{\alpha}', \beta', \gamma', 0) = \int \mathbf{D}^2\{\theta\}\mathbf{D}^2\{\alpha\}\mathbf{D}^2\{\beta\}\mathbf{D}^2\{\gamma\}$$
$$\times \exp\left\{\sum_{0\leqslant\tau<t}\left[\sum_{k=1}^{K}\theta_k^*(\tau+)\theta_k(\tau) + \sum_{r=1}^{M}\alpha_r^*(\tau+)\alpha_r(\tau) + \beta^*(\tau+)\beta(\tau) + \gamma^*(\tau+)\gamma(\tau)\right] - \mathrm{i}\sum_{k=1}^{K}\int_0^t \mathrm{d}\tau\,\overline{\Omega}_k\theta_k^*(\tau+)\theta_k(\tau)\right.$$
$$- \mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\,\Omega_r\alpha_r^*(\tau+)\alpha_r(\tau) - \frac{\mathrm{i}\omega}{2}\int_0^t \mathrm{d}\tau[\gamma^*(\tau+)\gamma(\tau) - \beta^*(\tau+)\beta(\tau)]$$
$$- \mathrm{i}\sum_{k=1}^{K}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\; g_{kr}(\alpha_r^*(\tau+) + \alpha_r(\tau))(\theta_k^*(\tau+) + \theta_k(\tau))$$
$$\left. - \mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau(g_r^*\alpha_r^*(\tau+) + g_r\alpha_r(\tau))(\gamma^*(\tau+)\beta(\tau) + \beta^*(\tau+)\gamma(\tau))\right\}. \tag{68}$$

Here the notations are obvious generalizations from the previous section to include the $K$ modes of the reservoir oscillators $b^\dagger, b$:

$$\mathbf{D}^2\{\boldsymbol{\theta}\} = \prod_{k=1}^{K} D^2\{\theta_k\} \tag{69}$$

$$\mathbf{D}^2\{\boldsymbol{\alpha}\} = \prod_{r=1}^{M} D^2\{\alpha_r\}. \tag{70}$$

The effective difference between (68) and (15) lies in the following term:

$$\begin{aligned}
&\exp\left\{-\mathrm{i}\sum_{k=1}^{K}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ g_{kr}(\alpha_r^*(\tau+)+\alpha_r(\tau))(\theta_k^*(\tau+)+\theta_k(\tau))\right\}\\
&\quad=\prod_{k=1}^{K}\exp\left\{-\mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ g_{kr}(\alpha_r^*(\tau+)+\alpha_r(\tau))(\theta_k^*(\tau+)+\theta_k(\tau))\right\}\\
&\quad=\prod_{k=1}^{K}\int D^2\{f_k\}\exp\left\{-\mathrm{i}\sum_{r=1}^{M}\int_0^t \mathrm{d}\tau\ f_k^*(\tau)g_{kr}(\alpha_r^*(\tau+)+\alpha_r(\tau))\right\}\\
&\qquad\times\exp\left\{\int_0^t \mathrm{d}\tau\ f_k(\tau)(\theta_k^*(\tau+)+\theta_k(\tau))\right\}
\end{aligned} \tag{71}$$

where we have introduced $K$ independent complex auxiliary fields.

Thus the propagator given by (68) is given in a manner analogous to (20) as

$$\begin{aligned}
&\widetilde{U}_1(\boldsymbol{\theta}^*,\boldsymbol{\alpha}^*,\beta^*,\gamma^*,t;\boldsymbol{\theta}',\boldsymbol{\alpha}',\beta',\gamma',0)\\
&\quad=\prod_{k=1}^{K}\int\int \mathbf{D}^2\{f\}\mathbf{D}^2\{f_k\}G_{1k}(\boldsymbol{\alpha}^*,t;\boldsymbol{\alpha}',0;[f^*,f_k^*])\\
&\qquad\times M_{1k}(\theta_k^*,t;\theta_k',0;[f_k])N_1(\beta^*,\gamma^*,t;\beta',\gamma',0;[f]).
\end{aligned} \tag{72}$$

Proceeding in a manner analogous to that outlined in the previous section, taking the $l=1$ term in $N_1$, and using the same approximation of $M_1^0$ for $M_1$, we obtain the amplitude of the propagator corresponding to the Hamiltonian (1) with the initial condition

$$\widetilde{U}_{ij}(t=0) = \exp\left\{\sum_{k=1}^{K}\theta_k^*\theta_k'\right\}\exp\left\{\sum_{r=1}^{M}\alpha_r^*\alpha_r'\right\}\delta_{ij} \tag{73}$$

in a matrix form as follows:

$$\begin{aligned}
\begin{bmatrix}\widetilde{U}_{00} & \widetilde{U}_{01}\\ \widetilde{U}_{10} & \widetilde{U}_{11}\end{bmatrix} &= \exp\left\{\sum_{k=1}^{K}\theta_k^*\theta_k'\,\mathrm{e}^{-\mathrm{i}\overline{\Omega}_k t}\right\}\exp\left\{\sum_{r=1}^{M}\alpha_r^*\alpha_r'\,\mathrm{e}^{-\mathrm{i}\Omega_r t}\right\}\\
&\quad\times\sum_{n=0}^{\infty}\left(\frac{\mathrm{i}\omega}{2}\right)^n\int_0^t \mathrm{d}\tau_n\int_0^{\tau_n}\mathrm{d}\tau_{n-1}\cdots\int_0^{\tau_2}\mathrm{d}\tau_1\\
&\quad\times\left\{\prod_{k=1}^{K}\frac{K_{\widetilde{\alpha}_k}(\overline{\Omega}_k)}{\sqrt{\widetilde{\alpha}_k^2+4\Theta_k\xi_k}}\exp\left\{-\frac{\mathrm{i}}{4}A_k\right\}\exp\left\{F_2^{(n)}\right\}\right.\\
&\quad\left.\times\begin{bmatrix}\cosh\left(\overline{\phi}_2^{(n)}\right) & \sinh\left(\overline{\phi}_2^{(n)}\right)\\ (-1)^n\sinh\left(\overline{\phi}_2^{(n)}\right) & (-1)^n\cosh\left(\overline{\phi}_2^{(n)}\right)\end{bmatrix}\right\}
\end{aligned} \tag{74}$$

where

$$\overline{\phi}_2^{(n)} = \phi_{21}^{(n)} - \sum_{k=1}^{K} (n_{1_k}(\overline{\Omega}_k)\theta_k^* \, \mathrm{e}^{-\mathrm{i}\overline{\Omega}_k t} + n_{1_k}\left(-\overline{\Omega}_k)\theta_k' + \frac{\mathrm{i}}{4} C_k\right) \tag{75}$$

with $A_k$, $C_k$, $n_{1_k}(\overline{\Omega}_k)$, $K_{\widetilde{\alpha}_k}(\overline{\Omega}_k)$, $\widetilde{\alpha}_k$, $\Theta_k$, $\xi_k$ given by (55), (56), (63), (41), (48), (51) and (52), respectively, with $g_{1r}$ replaced by $g_{kr}$, $\overline{\Omega}$ replaced by $\overline{\Omega}_k$, $\theta^*$ replaced by $\theta_k^*$, and $\theta'$ replaced by $\theta_k'$. It can be seen that (74) satisfies the initial condition given by (73). Even though the propagator (74) has been obtained in the form of quadratures, it cannot be evaluated in a closed form in the general case.

## 4. Quantum Brownian motion

We will first check our method on a simple model describing a damped harmonic oscillator [15]. This well-known example has been used by Caldeira–Leggett [2] to treat the problem of a quantum Brownian particle (an oscillator) interacting with an environment modelled by a set of oscillators. The Hamiltonian for this problem is

$$H = H_S + H_E + H_{SE} \tag{76}$$

where the Hamiltonian of the system is

$$H_S = \hbar\overline{\Omega} b^\dagger b \tag{77}$$

the Hamiltonian of the environment is

$$H_E = \sum_{r=1}^{M} \hbar\Omega_r a_r^\dagger a_r \tag{78}$$

and the interaction of the system with the environment is

$$H_{SE} = \sum_{r=1}^{M} \hbar g_{1r}\left(a_r^\dagger + a_r\right)(b^\dagger + b). \tag{79}$$

Now (76) is the same as (7) in the absence of the spin-1/2 component, and hence we have the propagator $K_1$ for the Hamiltonian (76) as (cf (20))

$$K_1(\theta^*, \boldsymbol{\alpha}^*, t; \theta', \boldsymbol{\alpha}', 0) = \int \mathbf{D}^2\{f_1\} G_1(\boldsymbol{\alpha}^*, t; \boldsymbol{\alpha}', 0; [f_1^*]) M_1(\theta^*, t; \theta', 0; [f_1]) \tag{80}$$

where $G_1$ stands for the propagator of

$$H_{G_1} = \hbar \sum_{r=1}^{M} \left[\Omega_r \alpha_r^* \frac{\partial}{\partial\alpha_r^*} + f_1^*(t) g_{1r}(\alpha_r^* + \alpha_r)\right] \tag{81}$$

$M_1$ stands for the propagator of

$$H_{M_1} = \hbar\left[\overline{\Omega}\theta^* \frac{\partial}{\partial\theta^*} + \mathrm{i} f_1(t)(\theta^* + \theta)\right]. \tag{82}$$

We now write the density matrix in the holomorphic representation as

$$\langle\theta_1\boldsymbol{\alpha}_1|\rho(t)|\theta_2\boldsymbol{\alpha}_2\rangle = \int \mathrm{d}\mu(\theta_1')\,\mathrm{d}\mu(\theta_2')\,\mathrm{d}\mu(\boldsymbol{\alpha}_1')\,\mathrm{d}\mu(\boldsymbol{\alpha}_2')\langle\theta_1\boldsymbol{\alpha}_1|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|\theta_1'\boldsymbol{\alpha}_1'\rangle$$
$$\times \langle\theta_1'\boldsymbol{\alpha}_1'|\rho(0)|\theta_2'\boldsymbol{\alpha}_2'\rangle\langle\theta_2'\boldsymbol{\alpha}_2'|\mathrm{e}^{\mathrm{i}Ht/\hbar}|\theta_2\boldsymbol{\alpha}_2\rangle \tag{83}$$

where $\mathrm{d}\mu(\theta_1') = \mathrm{e}^{-\theta_1'\theta_1'^*}\,\mathrm{d}^2\theta_1'/\pi$ and so on, and is the measure used in the completeness relation. The vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_M)$ denotes the reservoir coordinates while $\theta$ denotes the system coordinate. To obtain the reduced density matrix of the system alone, the reservoir coordinates are traced out and we have

$$\begin{aligned}\langle\theta_1|\rho(t)|\theta_2\rangle &= \int d\mu(\boldsymbol{\alpha}_1)\langle\theta_1\boldsymbol{\alpha}_1|\rho(t)|\theta_2\boldsymbol{\alpha}_1\rangle \\ &= \int d\mu(\theta_1')\,d\mu(\theta_2')\,d\mu(\boldsymbol{\alpha}_1')\,d\mu(\boldsymbol{\alpha}_2')\,d\mu(\boldsymbol{\alpha}_1) \\ &\quad\times K_{10}(\theta_1^*,\boldsymbol{\alpha}_1^*,t;\theta_1',\boldsymbol{\alpha}_1',t)K_{10}^*(\theta_2^*,\boldsymbol{\alpha}_1^*,t;\theta_2',\boldsymbol{\alpha}_2',0)\langle\theta_1'\boldsymbol{\alpha}_1'|\rho(0)|\theta_2'\boldsymbol{\alpha}_2'\rangle\end{aligned} \tag{84}$$

where $K_{10}$ is obtained from the propagator $K_1$ in (80) with the approximation of $M_1^0$ for $M_1$ as in (31), and evaluated using the procedure outlined in section 2.

If we assume that initially the system and reservoir density matrices were uncorrelated, we have

$$\langle\theta_1'\boldsymbol{\alpha}_1'|\rho(0)|\theta_2'\boldsymbol{\alpha}_2'\rangle = \rho_A(\theta_1'^*,\theta_2',0)\rho_B(\boldsymbol{\alpha}_1'^*,\boldsymbol{\alpha}_2',0) \tag{85}$$

where

$$\rho(0) = \rho_A(0)\rho_B(0) \tag{86}$$

with $\rho_A(0)$ being the initial system density matrix and $\rho_B(0)$ the initial reservoir density matrix. Let the system be initially in a pure coherent state $|\overline{\alpha}\rangle$ so that

$$\rho_A(\theta_1'^*,\theta_2',0) = e^{\theta_1'^*\overline{\alpha}}\,e^{\overline{\alpha}^*\theta_2'}. \tag{87}$$

For the initial reservoir density matrix, we assume an ensemble of harmonic oscillators at equilibrium at a temperature $T$, given by

$$\rho_B(\alpha_1'^*,\alpha_2',0) = \prod_{r=1}^{M} e^{-\hbar\Omega_r/2k_BT}\exp\{e^{-\hbar\Omega_r/k_BT}\alpha_{1r}'^*\alpha_{2r}'\} \tag{88}$$

where $k_B$ is the Boltzmann constant.

Now we substitute (85), (87), (88) in (84) to get the reduced density matrix of the system in a Gaussian form. In order to express the coefficients of the Gaussian in a simple form, we set $\overline{\Omega}$ equal to zero. This enables us to write the normalized reduced density matrix as

$$\rho(\theta_1^*,\theta_2,t) = C'\frac{1}{\sqrt{\overline{\alpha}_1\overline{\alpha}_2}}\exp\left\{-\frac{1}{(2\overline{\alpha}_1\overline{\alpha}_2)}[2y\theta_1^*\theta_2 - z\theta_2^2 - z^*\theta_1^{*2} - 2n^*\theta_2 - 2n\theta_1^*] + \theta_1^*\theta_2\right\} \tag{89}$$

where

$$C' = \frac{\sqrt{y^2-|z|^2}}{\sqrt{\overline{\alpha}_1\overline{\alpha}_2}}\exp\left\{-\frac{1}{(\overline{\alpha}_1\overline{\alpha}_2[y^2-|z|^2])}\left[\frac{z^*n^{*2}}{2} + y|n|^2 + \frac{zn^2}{2}\right]\right\} \tag{90}$$

with

$$\overline{\alpha}_1 = 1 + 2[\overline{G} - \overline{C}^*] \tag{91}$$

$$\overline{\alpha}_2 = 1 - \frac{z}{\overline{\alpha}_1} \tag{92}$$

$$z = 2\overline{\alpha}_1(\overline{C} - \overline{G}^*) + \overline{A}^2 \tag{93}$$

$$y = \overline{\alpha}_1\overline{\alpha}_2 - \overline{A} \tag{94}$$

$$n = \overline{\alpha}_1^*\overline{\alpha} + \overline{A}\overline{\alpha}^* \tag{95}$$

$$\overline{A} = \sum_{r=1}^{M}\coth\left(\frac{\hbar\Omega_r}{2k_BT}\right)(g_{1r})^2(\overline{S_r^*S_r}) \tag{96}$$

$$\overline{C} = \sum_{r=1}^{M} \frac{\mathrm{e}^{-\hbar\Omega_r/2k_BT}}{2\sinh\left(\frac{\hbar\Omega_r}{2k_BT}\right)} \mathrm{e}^{-\mathrm{i}\Omega_r t}(g_{1r})^2(\overline{S}_r^*)^2 \tag{97}$$

$$\overline{G} = \sum_{r=1}^{M} \frac{g_{1r}^2}{\Omega_r}(\overline{S}_r - \mathrm{i}t) \tag{98}$$

$$\overline{S}_r = \frac{1}{\Omega_r}(1 - \mathrm{e}^{-\mathrm{i}\Omega_r t}). \tag{99}$$

The reduced density matrix (89) for the system has the same form as known in the literature, e.g., [15].

## 5. An explicit evaluation

In this section we will explicitly evaluate the propagator for the following Hamiltonian with one spin and a dynamically coupled single-mode field:

$$H = \tfrac{1}{2}\hbar\omega\sigma_z + \hbar g_1(a^\dagger + a)(b^\dagger + b) + \hbar g(a^\dagger + a)\sigma_x. \tag{100}$$

The Hamiltonian (100) is a simplified version of the Hamiltonian (7), with $M = 1$, $g_{1r} \equiv g_1$ (real) and $g_r = g_r^* \equiv g$ (real).

We write the propagator of the bosonized form of this Hamiltonian, analogous to (20), as

$$U_2 = \int\int \mathbf{D}^2\{f\}\mathbf{D}^2\{f_1\}G_1M_1N_1. \tag{101}$$

Here

$$G_1 = G_{1a}G_{1b}G_{1c} \tag{102}$$

where

$$G_{1a} = \exp\left\{\alpha^*\alpha' - \mathrm{i}g(\alpha^* + \alpha')\int_0^t \mathrm{d}\tau\, f^*(\tau) - \frac{g^2}{2}\left(\int_0^t \mathrm{d}\tau\, f^*(\tau)\right)^2\right\} \tag{103}$$

$$G_{1b} = \exp\left\{-\mathrm{i}(\alpha^* + \alpha')g_1\int_0^t \mathrm{d}\tau\, f_1^*(\tau) - \frac{g_1^2}{2}\left(\int_0^t \mathrm{d}\tau\, f_1^*(\tau)\right)^2\right\} \tag{104}$$

$$G_{1c} = \exp\left\{-gg_1\int_0^t \mathrm{d}\tau\left(\int_0^t \mathrm{d}\tau' f_1^*(\tau')\right) f^*(\tau)\right\}. \tag{105}$$

From (29),

$$\begin{aligned} M_1 &= \exp\left\{\theta^*\theta' + (\theta^* + \theta')\int_0^t \mathrm{d}\tau\, f_1(\tau) + \frac{1}{2}\left(\int_0^t \mathrm{d}\tau\, f_1(\tau)\right)^2\right\} \\ &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\lambda \exp\left\{-\frac{1}{2}(\theta^{*2} + \theta'^2) - \frac{\lambda^2}{2} + \lambda(\theta^* + \theta') + \lambda\int_0^t \mathrm{d}\tau\, f_1(\tau)\right\}. \end{aligned} \tag{106}$$

This is done in order to facilitate the use of the $\delta$-functional identity (18). $N_1$ here is written not with $Q$ given in the form of a series as in (37), but in a product form as

$$Q(t) = \mathcal{T}\exp\left[-\mathrm{i}\int_0^t \mathrm{d}\tau\left(\frac{\omega}{2}\sigma_z + \mathrm{i}f(\tau)\sigma_x\right)\right] \tag{107}$$

where $\mathcal{T}$ denotes time ordering.

Now substituting (102) and (106) into (101), we have

$$\begin{aligned} U_2 = \int \mathbf{D}^2\{f\} G_{1a} \left( \int \mathbf{D}^2\{f_1\} G_{1b} G_{1c} M_1 \right) N_1 \\ = \frac{1}{\sqrt{2\pi}} \exp\{\alpha^* \alpha'\} \int \mathbf{D}^2\{f\} \int \mathrm{d}\lambda \exp\left\{ -\frac{1}{2}(\theta^{*2} + \theta'^2) \right. \\ \left. - \frac{\lambda^2}{2} + \lambda(\theta^* + \theta') - \mathrm{i}(\alpha^* + \alpha') g_1 \lambda t - \frac{g_1^2}{2} \lambda^2 t^2 \right\} \\ \times \exp\left\{ -\mathrm{i} g(\alpha^* + \alpha') \int_0^t \mathrm{d}\tau\, f^*(\tau) - g g_1 \lambda t \int_0^t \mathrm{d}\tau\, f^*(\tau) \right. \\ \left. - \frac{g^2}{2} \left( \int_0^t \mathrm{d}\tau\, f^*(\tau) \right)^2 \right\} N_1. \end{aligned} \tag{108}$$

Now the terms with $f^*(\tau)$ can be written as

$$\begin{aligned} \exp\left\{ -\frac{1}{2} g^2 \left( \int_0^t \mathrm{d}\tau\, f^*(\tau) \right)^2 - g\left[ \mathrm{i}(\alpha^* + \alpha') + g_1 \lambda t \right] \int_0^t \mathrm{d}\tau\, f^*(\tau) \right\} \\ = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\lambda_1 \exp\left\{ \frac{1}{2}(\mathrm{i}(\alpha^* + \alpha') + g_1 \lambda t)^2 - \frac{\lambda_1^2}{2} - \mathrm{i}\lambda_1 g \int_0^t \mathrm{d}\tau\, f^*(\tau) \right. \\ \left. + (\alpha^* + \alpha')\lambda_1 - \mathrm{i} g_1 \lambda \lambda_1 t \right\}. \end{aligned} \tag{109}$$

Expanding $N_1$ and taking the $l = 1$ term as before with $Q(t)$ given by (107), using (109) in (108), and applying the $\delta$-functional identity (18) gives us the propagator $\overline{U}_2$ of the Hamiltonian (100) as

$$\begin{aligned} \overline{U}_2 = \frac{1}{2\pi} \int \mathrm{d}\lambda \int \mathrm{d}\lambda_1 \exp\left[ -\mathrm{i}\left( \frac{\omega}{2}\sigma_z + \lambda_1 g \sigma_x + g_1 \lambda \lambda_1 \right) t \right] \exp\left\{ -\frac{1}{2}(\alpha^{*2} + \alpha'^2) \right. \\ \left. - \frac{1}{2}(\theta^{*2} + \theta'^2) + \lambda(\theta^* + \theta') + \lambda_1(\alpha^* + \alpha') - \frac{1}{2}(\lambda^2 + \lambda_1^2) \right\}. \end{aligned} \tag{110}$$

This can now be expanded in the power series of the matrices $\sigma_x$ and $\sigma_z$ to get the propagator in a matrix form.

It may be noted that Papadopoulos [9] has explicitly worked out the propagator for the Hamiltonian

$$H = \tfrac{1}{2}\hbar\omega\sigma_z + \hbar g(a^\dagger + a)\sigma_x.$$

We have the same form of eigenfunction expansion for the propagator with the eigenfunctions now having additional terms corresponding to the new variables, as well as a time-dependent term originating from the dynamics introduced into the Hamiltonian (100) by the term $\hbar g_1(a^\dagger + a)(b^\dagger + b)$. The result of Papadopoulos can easily be found from (110) by setting the appropriate terms to zero.

## 6. Summary

We have obtained the propagator for a very general Hamiltonian which can be used to describe a two-level atom interacting with a multimode radiation field (without the rotating-wave approximation) in a cavity with the cavity field linked to a reservoir. We proceeded by bosonizing a simpler Hamiltonian, namely, one having only one reservoir mode, and evaluated the propagator using the coherent state path integral method. Following Papadopoulos [9], the evaluation has been facilitated by functional integral averages over two complex auxiliary fields against Gaussian measures of the fields. The functional averaging over one auxiliary field is done exactly while the functional averaging over the second auxiliary field is done in a reasonable approximate manner. The propagator obtained by Papadopoulos [9] for the single spin–Bose system is reproduced satisfactorily in the appropriate limit. The result is then generalized to obtain the propagator for the original Hamiltonian with many modes of the thermal reservoir. The propagator is explicitly evaluated for a simplified version of the system with one spin and a dynamically coupled single-mode field.

The results are also checked on the known problem of quantum Brownian motion. We take the case where the system oscillator is initially in a pure coherent state, and the system and the reservoir are initially uncorrelated. We find a Gaussian form of the reduced density matrix describing the system alone. As opposed to a 'macroscopic' approach to such problems of open quantum systems in which a precise description of every degree of freedom is not used, our approach is a 'microscopic' one in that we first solve for the propagator of the Hamiltonian and the propagator can then be used to obtain the reduced density matrix of the system of interest.

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